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ON CLOSED-FORM ANALYTICAL SOLUTION FOR CIRCULAR DUCT, THERMALLY DEVELOPING SLUG FLOW UNDER MIXED BOUNDARY CONDITION

V. P. TYAGI and K. M. NIGAM
Department of Mathematics, Indian Institute of Technology, Bombay—400076, India

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Abstract—The present work deals with the problem of steady-state heat transfer in slug flow in a circular pipe with constant physical properties and negligible axial heat conduction. Use both of the Laplace transform with respect to the axial co-ordinate and of Galerkin's technique in the transform domain results in an approximate analytical solution. The calculations have shown that the second approximation thus obtained agrees well with the known exact analytical solution in the form of a series.

NOMENCLATURE

Bi, Biot number, same definition as in [1];
J_n, first kind Bessel function of order n;
Pe, Peclet number (heat-transfer case), same definition as in [1];
T, local temperature in slug flow;
T_a, ambient temperature;
T_0, initial temperature of slug flow;
s, Laplace transform parameter;
θ, $(T-T_a)/(T_0-T_a)$;
ρ, radial coordinate divided by tube radius;
ξ, axial coordinate divided by tube radius.

Subscript
m, mixed-mean value.

INTRODUCTION

RECENTLY, in connection with a problem of steady state heat transfer, with mixed type thermal boundary condition; constant physical properties; negligible axial heat conduction and uniform initial temperature, in slug flow in a pipe of circular cross-section, Golos [1] has considered the system of non-dimensional equations:

$$\frac{\partial^2\theta}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial\theta}{\partial\rho}-Pe\frac{\partial\theta}{\partial\xi}=0 \tag{1}$$

$$\theta(\rho,0)=1; \quad \theta(\rho,\infty)=0 \tag{2}$$

$$\left.\frac{\partial\theta}{\partial\rho}\right|_{\rho=0}=0; \quad \left[\frac{\partial\theta}{\partial\rho}+Bi\theta\right]_{\rho=1}=0 \tag{3}$$

and has given an approximate solution, based on restricted variation principle.

The exact solution of this system of equations can be found in [2, 3] and is given by, as cited in [1],

$$\theta=2Bi\sum_{n=1}^{\infty}\frac{J_0(\zeta_n\rho)}{(Bi^2+\zeta_n^2)J_0(\zeta_n)}\exp\left(-\frac{\zeta_n^2}{Pe}\xi\right) \tag{4}$$

where ζ_n are the roots of the transcendental equation

$$\zeta_n J_1(\zeta_n)-BiJ_0(\zeta_n)=0. \tag{5}$$

The definitions of the foregoing symbols and cylindrical polar coordinates with origin at centre of thermal entry section are the same as in [1].

However, the above-mentioned approximate solution, given by Golos [1], involves a function of ξ which is not determinable analytically. In the present study we give an approximate closed-form solution, which is analytically fully-determined in terms of both ρ and ξ and is superior to that of Golos. Also, the present numerical computations for exact solution would be superior to the earlier one [1].

PRESENT ANALYSIS

Using Laplace transform of θ with respect to axial co-ordinate, i.e.

$$\bar{\theta}=\int_0^\infty \theta\exp(-s\xi)\,d\xi \tag{6}$$

equations (1)–(3) reduce to

$$\frac{d^2\bar{\theta}}{d\rho^2}+\frac{1}{\rho}\frac{d\bar{\theta}}{d\rho}-Pe(s\bar{\theta}-1)=0 \tag{7}$$

$$\left.\frac{d\bar{\theta}}{d\rho}\right|_{\rho=0}=0; \quad \left[\frac{d\bar{\theta}}{d\rho}+Bi\bar{\theta}\right]_{\rho=1}=0. \tag{8}$$

We apply Galerkin technique [4] to the system of equations (7) and (8) and calculate only first and second Galerkin approximations, $\bar{\theta}_{(1)}$ and $\bar{\theta}_{(2)}$, for $\bar{\theta}$. Let the inverse Laplace transforms of $\bar{\theta}_{(1)}$ and $\bar{\theta}_{(2)}$ be denoted by $\theta_{(1)}$ and $\theta_{(2)}$, respectively. Then $\theta_{(1)}$ and $\theta_{(2)}$ would be the first and second Galerkin approximations for θ.

The calculation of $\bar{\theta}_{(1)}$ and that of $\bar{\theta}_{(2)}$ are very simple, if one succeeds in searching for an infinite family of functions of ρ satisfying the requirements that the functions satisfy (8), possess first two continuous derivatives and constitute a linearly independent set of functions on $0\leqslant\rho\leqslant 1$. We introduce the family of functions

$$\Phi_n=1-\frac{Bi}{2n+Bi}\rho^{2n}, \quad (n=1,2,3,\ldots), \tag{9}$$

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satisfying these necessary requirements. It is noteworthy that these functions are algebraic polynomials. Therefore, the set of functions Φ_n, given by (9), is complete in terms of Weierstrass theorem.

From the family of functions in (9), only Φ_1 is employed to calculate $\bar{\theta}_{(1)}$, and Φ_1 and Φ_2 are employed to calculate $\bar{\theta}_{(2)}$. Therefore, the expression of $\theta_{(1)}$ involves Φ_1 and that of $\theta_{(2)}$ involves Φ_1 and Φ_2.

Without showing mathematical steps, we present our final results for $\theta_{(1)}$ and $\theta_{(2)}$ as follows:

$$\left.\begin{aligned}\theta_{(1)} = \frac{3(Bi+2)(Bi+4)}{2(Bi^2+6Bi+12)}\left(1-\frac{Bi\rho^2}{2+Bi}\right)\exp\left(A_1\frac{\xi}{Pe}\right)\end{aligned}\right\} \quad (10)$$

$$A_1 = -6Bi(Bi+4)/(Bi^2+6Bi+12)$$

$$\left.\begin{aligned}\theta_{(2)} &= \left(1-\frac{Bi\rho^2}{Bi+2}\right) \times \sum_{n=1}^{2}\frac{8(Bi+2)\{20(Bi+12)-s_n(Bi+14)\}}{2A_2s_n+A_3} \times \exp\left(s_n\frac{\xi}{Pe}\right) \\ &+\left(1-\frac{Bi\rho^4}{Bi+4}\right)\sum_{n=1}^{2}\frac{10(Bi+4)(Bi+12)s_n}{2A_2s_n+A_3} \times \exp\left(s_n\frac{\xi}{Pe}\right) \\ A_2 &= 3Bi^2+48Bi+256; \\ A_3 &= 128(Bi^2+14Bi+30)\end{aligned}\right\} \quad (11)$$

where s_1 and s_2 are the roots of the quadratic equation

$$A_2s^2+A_3s+640Bi(Bi+12) = 0. \quad (12)$$

Since the exact analytical expressions of s_1 and s_2 can be given immediately, the closed-form results (10) and (11) are analytically fully-determined.

If the present method is convergent, then the second order approximation, $\theta_{(2)}$, should be superior to the first order one, $\theta_{(1)}$. This will be examined in the following section.

Table 1. Comparison of various temperature solutions for a case of liquid sodium

ρ	$\theta^{(10)}$	$\theta_{(2)}$	θ_{Golos}
	$\xi/\xi_r = 0{\cdot}17$		
0·00	0·9990	0·9993	1·0000
0·35	0·9986	0·9987	1·0000
0·70	0·9772	0·9761	1·0000
0·80	0·9583	0·9579	0·9923
0·90	0·9297	0·9308	0·9690
1·00	0·8905	0·8926	0·9302
	$\xi/\xi_r = 0{\cdot}65$		
0·00	0·9651	0·9674	1·0000
0·15	0·9616	0·9635	1·0000
0·30	0·9510	0·9518	1·0000
0·40	0·9396	0·9396	0·9970
0·50	0·9243	0·9237	0·9880
0·60	0·9049	0·9040	0·9730
0·70	0·8811	0·8804	0·9510
0·80	0·8528	0·8526	0·9240
0·90	0·8201	0·8204	0·8900
1·00	0·7830	0·7836	0·8510
	$\xi/\xi_r = 1{\cdot}00$		
0·00	0·9121	0·9129	1·0000
0·15	0·9080	0·9086	0·9950
0·30	0·8956	0·8959	0·9820
0·40	0·8826	0·8827	0·9680
0·50	0·8659	0·8658	0·9500
0·60	0·8454	0·8452	0·9280
0·70	0·8212	0·8210	0·9020
0·80	0·7933	0·7932	0·8720
0·90	0·7619	0·7619	0·8380
1·00	0·7271	0·7273	0·8000
	$\xi/\xi_r = 2{\cdot}00$		
0·00	0·7547	0·7547	0·8230
0·15	0·7510	0·7510	0·8190
0·30	0·7398	0·7398	0·8080
0·40	0·7283	0·7283	0·7960
0·50	0·7136	0·7136	0·7820
0·60	0·6959	0·6959	0·7640
0·70	0·6752	0·6752	0·7220
0·80	0·6517	0·6517	0·7170
0·90	0·6256	0·6256	0·6890
1·00	0·5969	0·5969	0·6580

DISCUSSIONS

Taking numerical values of first six ζ_n from [2] and using first six terms of (4), Golos has compared his approximate solution with exact solution. The comparison has been made for the problem of temperature distribution in liquid sodium flow with Peclet number $Pe = 22600$, Biot number $Bi = 0{\cdot}5$ and $\xi_r = 4972$, where ξ_r denotes the axial location at which thermal boundary layers meet. For this problem, we compare $\theta_{(2)}$ with exact solution (4) and with the approximate solution obtained in [1].

We calculated more than six roots of (5) to observe numerical convergence of (4) and found that it was accurate enough to perform computations on the basis of first ten terms of (4) for a range of Bi including $Bi = 0{\cdot}5$. The computed results are assembled in Table 1, where $\theta^{(10)}$ represents exact solution based on the first ten terms of (4). The entries in the fourth column have been picked up from [1]. It is seen that the difference between $\theta_{(2)}$ and $\theta^{(10)}$ is everywhere smaller than that between θ_{Golos} and $\theta^{(10)}$. Therefore, the present solution $\theta_{(2)}$ may be said to be superior to the solution of Golos.

Variation in the shape of developing temperature profile with variation in the value of Bi is shown in Fig. 1. The curves have been drawn by computing the solution $\theta_{(2)}$. It is noteworthy that the qualitative picture given by Fig. 1 is feasible.

Mixed-mean temperature is an important quantity to be investigated. It can be calculated very easily from each of (10) and (11) and is, therefore, shown directly in Table 2. It is seen that mixed-mean temperature decreases as ξ and Bi increase. This shows that the present solutions, $\theta_{(1)m}$ and $\theta_{(2)m}$, are qualitatively satisfactory. It is seen, from the second and fourth columns, that the absolute difference $|\theta_{(1)m}-\theta_m^{(10)}|$ is

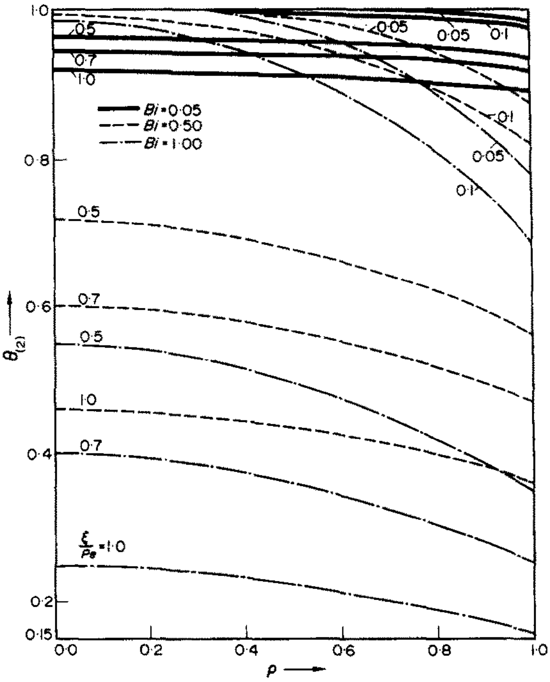

FIG. 1. Axially varying temperature profiles at various values of *Bi*.

small and differs insignificantly from $|\theta_{(2)m}-\theta_m^{(10)}|$ for $Bi \leqslant 0{\cdot}1$. The difference between $\theta_{(2)m}$ and $\theta_m^{(10)}$, from the third and fourth columns, is very small and decreases as ξ increases. As is seen that $|\theta_{(2)m}-\theta_m^{(10)}|$ is smaller than $|\theta_{(1)m}-\theta_m^{(10)}|$, $\theta_{(2)}$ is superior to $\theta_{(1)}$. This indicates that, in the case of present problem, Galerkin method is convergent. That is, the third order solution is expected to be superior to $\theta_{(2)}$ and so on. The rigorous proof for convergence is not given here, as it involves certain abstract concepts of "pure mathematics".

The family of functions Φ_n, introduced by (9), is also suitable to cases of Newtonian and non-Newtonian fluids and analogous problems of flat conduit geometry.

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Table 2. Comparison of various mixed-mean temperature solutions at several values of *Bi*

ξ/Pe	$\theta_{(1)m}$	$\theta_{(2)m}$	$\theta_m^{(10)}$	$\theta_{(1)m}$	$\theta_{(2)m}$	$\theta_m^{(10)}$
		$Bi = 0{\cdot}1$			$Bi = 1{\cdot}0$	
0·000	0·9998017	0·9999872	0·9999998	0·9868421	0·9989142	0·9999853
0·005	0·9988270	0·9989985	0·9990054	0·9790819	0·9899986	0·9905198
0·010	0·9978532	0·9980118	0·9980155	0·9713828	0·9812029	0·9814567
0·050	0·9900969	0·9901811	0·9901793	0·9119295	0·9158468	0·9156932
0·100	0·9804863	0·9805231	0·9805225	0·8427036	0·8433028	0·8432655
0·200	0·9615440	0·9615485	0·9615487	0·7196180	0·7184993	0·7185163
0·300	0·9429677	0·9429652	0·9429653	0·6145039	0·6133573	0·6133648
0·400	0·9247502	0·9247464	0·9247464	0·5247548	0·5238155	0·5238174
0·500	0·9068847	0·9068807	0·9068807	0·4481089	0·4473841	0·4473843
1·000	0·8226030	0·8225994	0·8225994	0·2034790	0·2033472	0·2033470
		$Bi = 0{\cdot}5$			$Bi = 2{\cdot}0$	
0·000	0·9959016	0·9997032	0·9999963	0·9642857	0·9963370	0·9999416
0·005	0·9915033	0·9949786	0·9951334	0·9519671	0·9803020	0·9819871
0·010	0·9871244	0·9902989	0·9903763	0·9398059	0·9647802	0·9654707
0·050	0·9527821	0·9542592	0·9542174	0·8479454	0·8554646	0·8549704
0·100	0·9115295	0·9119817	0·9119685	0·7456414	0·7446335	0·7445717
0·200	0·8343052	0·8341268	0·8341308	0·5765730	0·5726402	0·5726994
0·300	0·7636234	0·7633584	0·7633556	0·4458395	0·4428114	0·4428281
0·400	0·6989297	0·6986719	0·6986727	0·3447489	0·3427742	0·3427769
0·500	0·6397168	0·6394886	0·6394888	0·2665797	0·2653914	0·2653897
1·000	0·4109217	0·4108146	0·4108146	0·0736968	0·0738526	0·0738522

SUR UNE SOLUTION ANALYTIQUE DU REGIME D'ETABLISSEMENT THERMIQUE EN ECOULEMENT RAMPANT DANS UN TUBE CIRCULAIRE AVEC DES CONDITIONS AUX LIMITES MIXTES

Résumé—Le présent travail traite du problème du transfert thermique stationnaire en écoulement rampant dans un tube circulaire avec propriétés physiques constantes et conduction thermique axiale négligeable.

L'utilisation de la transformation de Laplace par rapport à la coordonnée axiale et d'une méthode de Galerkin dans le domaine transformé permet d'obtenir une solution analytique approchée. Les calculs ont montré que la seconde approximation ainsi obtenue est en bon accord avec la solution analytique exacte connue sous forme de développement en série.

ÜBER EINE GESCHLOSSENE LÖSUNG FÜR THERMISCH ENTWICKELTE PFROPFENSTRÖMUNG IN KREISROHREN BEI GEMISCHTEN RANDBEDINGUNGEN

Zusammenfassung—Die vorliegende Arbeit beschäftigt sich mit der Frage des stationären Wärmeübergangs in einer Pfropfenströmung in einem Kreisrohr mit konstanten Stoffeigenschaften und bei vernachlässigbarer axialer Wärmeleitung.

Der Gebrauch sowohl der Laplace-Transformation unter Berücksichtigung der axialen Koordinate und der Technik von Galerkin im Transformationsbereich ergibt eine analytische Näherungslösung. Die Berechnungen haben gezeigt, daß die zweite Näherung, die man auf diese Weise erhält, gut überinstimmt mit der bekannten genauen analytischen Lösung in der Form einer Reihe.

АНАЛИТИЧЕСКОЕ РЕШЕНИЕ В ЗАМКНУТОЙ ФОРМЕ ДЛЯ ТЕПЛООБМЕНА В КРУГЛОЙ ТРУБЕ СО СТЕРЖНЕВЫМ ТЕЧЕНИЕМ ПРИ СМЕШАННЫХ ГРАНИЧНЫХ УСЛОВИЯХ

Аннотация — В работе рассматривается стационарный теплообмен в круговой трубе со стержневым потоком при постоянных физических свойствах и в пренебрежении теплопроводностью вдоль оси. В результате применения преобразования Лапласа по продольной координате и использования метода Галеркина в области изображений найдено приближенное решение в аналитической форме. Расчеты показали, что полученное таким образом второе приближение хорошо совпадает с известным точным аналитическим решением, имеющим вид ряда.